arXiv:hep-th/9703174v2 6 Jan 1998

Chern-Simons theory in 11 dimensions as a non-perturbative phase of M theory

Lee Smolin*

Center for Gravitational Physics and Geometry
Department of Physics
The Pennsylvania State University
University Park, PA, USA 1

Revised, December 30, 1997

ABSTRACT

A Chern-Simons theory in 11 dimensions, which is a piece of the 11 dimensional supergravity action, is considered as a quantum field theory in its own right. We conjecture that it defines a non-perturbative phase of M theory in which the metric and gravitino vanish. The theory is diffeomorphism invariant but not topological in that there are local degrees of freedom. Nevertheless, there are a countable number of momentum variables associated with relative cobordism classes of embeddings of seven dimensional manifolds in ten dimensional space. The canonical theory is developed in terms of an algebra of gauge invariant observables. We find a sector of the theory corresponding to a topological compactification in which the geometry of the compactified directions is coded in an algebra of functions on the base manifold. The diffeomorphism invariant quantum theory associated to this sector is constructed, and is found to describe diffeomorphism classes of excitations of three surfaces wrapping homology classes of the compactified dimensions.

* smolin@phys.psu.edu

1 Introduction

String theory has recently evolved in a fascinating direction, leading to evidence that it represents a class of perturbative expansions around vacuum states of a non-perturbative theory, whose nature remains unknown[1]. This conjectured non-perturbative theory has been called $\mathcal{M}$ theory[1, 2]. There is evidence that 11 dimensional supergravity plays an important role in its formulation, at the very least there may be a phase of this theory whose classical limit corresponds to 11 dimensional supergravity.

Despite some provocative suggestions[3], the nature of $\mathcal{M}$ theory at the fundamental, non-perturbative level remains unknown. A non-perturbative theory of quantum gravity must be one that relies on no background classical metric to give meaning to either its algebra of observables or perturbation expansions. Classical spacetime must emerge from a study of collective degrees of freedom that describe the critical behavior of such a theory, it cannot play a role in the formulation of the theory. But if geometry thus plays no fundamental role, the theory must be formulated entirely in algebraic and/or topological terms.

In the search for such a theory, two classes of results may offer useful hints. The first is topological quantum field theory, which shows that there are deep relationships between algebra, representation theory and topology[4, 6]. In its deepest formulation, in terms of the theory of tensor categories[5], $TQFT$ reveals new kinds of structures that may play a role in a non- perturbative formulation of a quantum theory of gravity. These structures are in fact closely related to conformal field theory[7].

Furthermore, the topological quantum field theories are based on finite dimensional representations of certain observable algebras. This is very good, as there are independent arguments from the Bekenstein bound[8] and the holographic hypothesis[9, 10] that tell us that any quantum theory of gravity must have a state space that decomposes into finite dimensional subspaces corresponding to measurements made on the boundaries of regions with finite surface area[11].

The simplest examples of the algebraic structures in $TQFT$ are spin networks[12] and quantum spin networks[13]. It is interesting that these label the difeomorphism invariant states of quantum general relativity[14] (or of any diffeomorphism invariant quantum field theory whose configuration space is based on a space of connections[15].) Quantum general relativity is now understood at the kinematical level (corresponding to spatially diffeomorphism invariant states[17]) where it has been found that the discrete and

combinatorial nature of the spin networks correspond to the discreteness of quantum geometry at the non-perturbative level[16, 18][1].

At the dynamical level, despite some non-trivial results[23, 24, 25, 26, 21, 22], it is not at all clear that quantum general relativity may have good critical behavior such as to allow the existence of a good continuum limit[27]. (That is the theory may have a status corresponding to random surface theory away from a critical point: it is well defined microscopically but has no interesting macroscopic behavior which may be described in terms of massless fields on a classical background.) In any case the search for such critical behavior need not rely on a quantization of the dynamics of classical general relativity[28].

Despite this, the results and methods discovered in the study of non-perturbative quantum general relativity may provide hints for the construction of a completely non-perturbative formulation of $\mathcal{M}$ theory. If one takes this point of view there are a number of possible starting points. One is to extend the spin network/TQFT picture to representations of algebras that play a role in string theory. Some results in this direction will be reported elsewhere, here I would like to describe some results from a more modest approach, which is to apply the methods of diffeomorphism invariant quantum field theory directly to supergravity in 11 dimensions[2].

In fact, what is studied here is only a part of that problem. If one sets the metric and gravitino field to zero, 11 dimensional supergravity[29] reduces to a Chern-Simons like theory based on a three form A_{abc}. The action of the theory is simply

$$S = \int_{\mathcal{M}_{11}} A \wedge F \wedge F \tag{1}$$

where $F = dA$ is a four form and $\mathcal{M}_{11}$ is an eleven dimensional manifold.

There are several reasons why it is useful to consider the quantization of this theory before taking on the full eleven dimensional supergravity. Even if 11 dimensional supergravity corresponds to no good quantum theory, this theory may be of interest, as it may yield a completely non-perturbative description of the extended objects such as D branes that play a crucial role in string theory[30]. For example, many of the results concerned with the entropies of extremal and near-extremal black holes come down to counting

[1]For a longer version of this argument see [19] The results of [17, 14, 18, 23] have also been reformulated in the language of rigorous mathematical quantum field theory[20, 21, 22].

[2]A new formulation of 11 dimensional supergravity in terms of new variables analogous to the Ashtekar variables has been given in [40].

topological embeddings and intersections of D branes in the compactified manifolds[31]. It seems likely that there must be a non-perturbative analogue of this in which these countings reduce to the topology and combinatorics of diffeomorphism invariant states. If so, it is likely that these have to do with the non-perturbative excitations of the A_{abc} field. A study of these in the absence of the metric and gravitino may then yield insights into $\mathcal{M}$ theory.

In any case, it is unlikely that anything interesting can come from the quantization of the metric parts of 11 dimensional supergravity unless it has a more interesting formulation analogous to the JSS[32] and CDJ[33] actions of the four dimensional theory. Although there has recently been progress towards that goal [40] by Melosch and Nicolai, that formulation is still somewhat complicated. It is then interesting to ask whether structures associated with the pure A_{abc} sector might give hints for how to reformulate 11 dimensional supergravity in a manner which could lead to more progress with the quantization.

But perhaps the best reason for considering the theory (1) is that it may define a phase of $\mathcal{M}$ theory. In non-perturbative quantum general relativity we have learned that in those forms of the theory amenable to non-perturbative treatment the classical phase space is extended to include solutions in which the metric is degenerate, or vanishes altogether[16]. This seems to be the consequence of seeking to describe the theory directly in terms of a algebras of fields corresponding to the full geometry, and not just to waves moving on classical backgrounds. Although the standard form of the 11 dimensional supergravity action is not polynomial in the variables, one can investigate whether the action and equations of motion have a good limit in which one scales the one form frame fields e_a^I and the gravitino field Ψ_a as $e_a^I = te_a^I$ and $\Psi_a = t\Psi_a$ and then takes the limit $t \to 0$. The theory does have such a limit, which is given by (1). This suggests that in a non-perturbative treatment there should be a sector of the state space in which e_a^I and Ψ_a vanish[3]. But it may be expected that any non-perturbative theory corresponding to the supergravity action in 11 dimensions describes a phase of $\mathcal{M}$ theory. Thus, it is plausible that what is described in this paper is the a non-perturbative description of a phase of $\mathcal{M}$ theory.

In fact, we find here that the theory (1) has lots of non-trivial structure. Most significantly, there is a sector of solutions in which there is a natural

[3] Of course the limit breaks supersymmetry. But $\mathcal{M}$ theory must have some phases that break supersymmetry, otherwise it can have nothing to do with nature.

compactification in which the physics of the compactified dimensions is completely topological. As a result, the dynamical content of the compactified directions is entirely represented in terms of an algebra of fields on the uncompactified dimensions. This provides a clue that the wealth of phenomena associated with different compactifications may eventually be understood in a different way, which is completely algebraic and combinatoric.

We may note that the theory (1) has certain similarities to five dimensional Chern-Simons theory. That has been studied by [34, 35, 36] and some of the results can be extended directly to the present case. Another study of higher dimensional Chern-Simons theories, which contains some results about the eleven dimensional case which are complementary to those described here is presented in [39].

We first sketch the canonical formulation of the theory that follows from (1). In section 4 we introduce the observable algebra which plays the role of the loop algebra in quantum gravity and 3 dimensional Chern-Simons theory. In sections 5 and 6 we then restrict the theory to two sectors of its solution space where we can find the full observables algebra and carry out the quantization, which we do in section 7.

2 The classical theory

The equations of motion coming from the action (1) are,

$$F \wedge F = 0 \tag{2}$$

We may note that one class of solutions may be constructed as follows. Let us consider an 11 dimensional manifold $\mathcal{M}_{11}$ which is locally a product of a d dimensional manifolds $\mathcal{Z}$ and an $11-d$ dimensional manifold $\mathcal{R}$. These are coordinatized respectively by $z^i, i = 1, ..., d$ and y^α, for $\alpha = 1, ..., 11 - d$. Then on $\mathcal{M}$ we may choose coordinates $x^{\hat{a}}$, with $\hat{a} = 1, ..., 11$ which split as $x^{\hat{a}} = (z^i, y^\alpha)$. Let us consider a class of three forms A_{abc} such that

$$F_{\alpha\hat{b}\hat{c}\hat{d}} = 0 \tag{3}$$

Then the Bianchi identity $dF = 0$ implies that $\partial_\alpha F_{ijkl} = 0$. The space of such $F_{\hat{a}\hat{b}\hat{c}\hat{d}}$'s is parameterized by a closed form F_{ijkl} on the d dimensional manifold $\mathcal{Z}$. Clearly to be nontrivial $d \geq 4$. But if $d < 8$ we have a solution to (2). These are not[4] the most general solutions to (2). Other classes of

[4]Other classes of solutions are considered in [39].

solutions are considered in [39]. In this paper we will be primarily interested in solutions of the form (3) as they are associated to splittings of $\mathcal{M}_{11}$ into products of lower dimensional manifolds. In the maximal case we have $11 = 7 + 4$; this may be relevant for non-perturbative "compactifications" to four spacetime dimensions.

The theory has two kinds of gauge invariance, eleven dimensional diffeomorphism invariance, given locally by

$$\delta_v A = \mathcal{L}_v A \tag{4}$$

and an abelian gauge invariance,

$$\delta_\lambda A = d\lambda \tag{5}$$

where λ is a two form. However, λ's of the form, $\lambda = d\rho$, where ρ is a one form do not contribute to the gauge transformations. There are then $11 * 10/2 - 11 = 44$ degrees of freedom of gauge transformations. The counting of the degrees of freedom is subtle and requires the canonical analysis; the complete counting is carried out in [39] where it is found that in the general case there are 19 local degrees of freedom. Here we will study some reduced sectors, in which there are local degrees of freedom which are expressed as functions on lower dimensional manifolds.

3 The canonical theory

We now assume that $\mathcal{M}_{11} = R \times \mathcal{N}_{10}$ where $\mathcal{N}_{10}$ is a compact ten dimensional manifold. Locally we may split the coordinates, so that $x^{\hat{a}} = (x^0, x^a)$, where $a = 1, ...10$. From now on all objects are ten dimensional. The action decomposes as,

$$S = \int dx^0 \int_{\mathcal{N}_{10}} \left(A^0 \wedge F \wedge F - \dot{A} \wedge A \wedge F \right) \tag{6}$$

Here A and F are the pull backs of the corresponding forms to the spatial sections and A^0 is a two form, whose components in local coordinates are $A^0_{bc} = A_{0bc}$. The canonical momenta are,

$$\pi^{abc} = (A \wedge F)^{*abc} \tag{7}$$

This gives rise to a set of primary constraints, which are

$$C^{abc} = \pi^{abc} - (A \wedge F)^{*abc} = 0 \tag{8}$$

The momenta conjugate to A^0 vanish as well, which gives rise to a set of secondary constraints,

$$H^{ab} = (F \wedge F)^{*ab} = 0 \tag{9}$$

The action is then of the form,

$$S = \int dx^0 \int_{\mathcal{N}_{10}} \left(\pi^{abc} \dot{A}_{abc} - A^0_{ab} H^{ab} \right) \tag{10}$$

We first may separate out a set of first class constraints that generate the abelian gauge transformations. These are,

$$G^{bc} = \partial_a C^{abc} \approx \partial_a \pi^{abc} \approx 0 \tag{11}$$

where $\approx$ means the equality holds on the constraint surface. It would be interesting to carry out a full analysis of the constraint algebra, but this has not yet been done. Because of this I will focus on a sector of the theory below.

On the constraint surface $C^{abc} = 0$ we may write the symplectic form

$$\omega(\delta_1 A, \delta_2 \pi) = \int_{\mathcal{N}} \delta_1 A_{abc} \delta_2 \pi^{abc} \tag{12}$$

as,

$$\omega(\delta_1 A, \delta_2 A) = \int_{\mathcal{N}} \delta_1 A \wedge \delta_2 A \wedge F \tag{13}$$

ω may be inverted for generic F_{abcd}, not subject to the constraint $F \wedge F = 0$. To see this, one may view $F^{*abcdef}$ as a metric on the space of three forms. Let indices A, B, C represent the 120 three form indices abc. Then F^{*AB} is a symmetric metric that may be inverted generically to find $\rho_{AB}(A)$ such that $F^{*AB}\rho_{BC}(A) = \rho_{CB}(A)F^{*BA} = \delta^A_C = \delta^{abc}_{def}$. Then the kinematical Poisson brackets are

$$\{A(x)_{abc}, A(y)_{def}\} = \rho_{abcdef}(x)\delta^{10}(x, y) \tag{14}$$

Unfortunately, the matrix F^{*AB} is degenerate on the constraint surface $F \wedge F = 0$ and does not yield the physical Poisson brackets of the theory, except in special cases. The problem of inverting the symplectic form (13) in the presence of the constraint $F \wedge F = 0$ is, as far as I know, not solved in general; it is related to the problem of making a complete analysis of the constraints. Below I will discuss how this may be done in one sector of the theory.

4 Surface variables and algebra

To quantize the theory, we may try to follow the method that worked in lower dimensional Chern-Simons theory, gauge theories and general relativity and construct an algebra of gauge invariant variables associated with embeddings of submanifolds in $\mathcal{N}_{10}$ of various dimensions[5]. To begin with we construct a set of variables associated with three dimensional surfaces embedded in $\mathcal{N}_{10}$. For every such surface γ, we may define an observable

$$T[\gamma] \equiv e^{\int_\gamma A}. \tag{15}$$

Conjugate to this we have a momentum variable, associated with compact seven dimensional submanifolds S. This is

$$\pi[S] \equiv \int_S \pi^* \tag{16}$$

The Poisson bracket between them involves the oriented intersection number $I[\gamma, S]$ between the three and seven dimensional submanifolds.

$$\{T[\gamma], \pi[S]\} = I[\gamma, S] T[\gamma] \tag{17}$$

It is easy to see that on the constraint surface most of the momenta are not independent. Instead, let S' be cobordic to S relative to $\mathcal{N}$. This means that there is an eight dimensional submanifold R of $\mathcal{N}_{10}$ such that $\partial R = S \cup \tilde{S}'$ (where $\tilde{S}$ is S with the reversed orientation.)

Then we have,

$$\pi[S] - \pi[S'] \approx \int_S A \wedge F - \int_{S'} A \wedge F = \int_R F \wedge F \approx 0 \tag{18}$$

It follows that there is an independent momentum variable $\pi[S]$ associated to each relative cobordism class of seven manifolds S in $\mathcal{N}$.

At the same time, the theory has local degrees of freedom, as we may exhibit sets of solutions labeled by continuous parameters, as we described above. We now turn to a study of a sector of the theory in which we can see how the interplay of a finite number of momentum variables with continuous spaces of solutions works out.

[5] The quantization of antisymmetric tensor gauge fields in terms of surface observables was considered previously in [37, 38].

5 Topological sectors

We now consider several sector of solutions on which we will be able to construct the Poisson brackets. In this section we will discuss a sector of solutions that is topological, in the sense that there are a finite number of physical degrees of freedom. In the next section we will see that we can also have sectors with local degrees of freedom.

We will assume that locally $\mathcal{N}_{10} = \mathcal{Z}_d \times \mathcal{Y}_{10-d}$ for compact manifolds $\mathcal{Z}_d$ and $\mathcal{Y}_{10-d}$ of the indicated dimensions. We assume we have local coordinates $z^i, i = 1, ..., d$ on $\mathcal{Z}_d$ and $y^\alpha, \alpha = 1, ..., 10-d$ on $\mathcal{Y}_{10-d}$. Thus, locally the coordinates x^a on $\mathcal{N}_{10}$ can be split as $x^a = (z^i, y^\alpha)$. Globally, we will require that $\mathcal{N}_{10}$ is a bundle over $\mathcal{Z}_d$ fibered by $\mathcal{Y}_{10-d}$, with projection map π. We will restrict attention to a sector of solutions to the constraint (9) of the form,

$$F_{\alpha abc} = 0 \tag{19}$$

The Bianchi identities $dF = 0$ then imply $\partial_\alpha F_{ijkl} = 0$. This sector of the solution space is then labeled by closed 4-forms on $\mathcal{Z}_d$ together with a set of functions $\phi[\tilde{\gamma}](z)$ on $\mathcal{Z}_d$, which are defined as follows. Each function is labeled by the $\tilde{\gamma}$, which are the homology classes of a three manifold in $\mathcal{Y}_{10-d}$. Given a compact three manifold $\gamma^\alpha(\sigma)$ (with coordinates $\sigma^I, I = 1, 2, 3$) in $\mathcal{Y}_{10-d}$ we have a d parameter set of manifolds in $\mathcal{N}_{10}$, $\gamma^a(\sigma)(z) = (\gamma^\alpha(\sigma), z^i)$. Each of these are three surfaces embedded in the fiber over the point $z \in \mathcal{Z}$.

We then may define the functions

$$\phi[\tilde{\gamma}](z) \equiv T[\gamma(z)] \tag{20}$$

By (19) these are function only of the homology class of its embedding in $\mathcal{Y}_{10-d}$. Thus, on each $\mathcal{Y}_{10-d}$ fiber the degrees of freedom are topological.

In some cases we can find the Poisson brackets of these functions. To do this let us consider the behavior of the symplectic form (13) on this sector of solutions. ω is degenerate and block diagonal, the only non-zero entries are

$$\omega(\delta_1 A_{\alpha\beta\gamma}, \delta_2 A_{\delta\epsilon\phi}) = \int \delta_1 A_{\alpha\beta\gamma} \delta_2 A_{\delta\epsilon\phi} F^{*\alpha\beta\gamma\delta\epsilon\phi} \tag{21}$$

where $F^{*\alpha\beta\gamma\delta\epsilon\phi} = \epsilon^{\alpha\beta\gamma\delta\epsilon\phi ijkl} F_{ijkl}$. We will now concentrate on the simplest case, which is $d = 4$. We then have one non-trivial component of F_{ijkl}, which is

$$F_{ijkl} = \epsilon_{ijkl} \tilde{\Psi}(z) \quad with \quad \tilde{\Psi}(z) = \frac{1}{24} \epsilon^{mnop} \partial_m A_{nop} \tag{22}$$

We note that $\tilde{\Psi}$ is a density on $\mathcal{Z}$, which may be set to a constant by a four dimensional diffeomorphism. Thus there are no local degrees of freedom from the A_{ijk}. This can also be seen from counting, there are four A_{ijk} but these are eliminated by local gauge transformations and four dimensional diffeomorphisms (which are not independent.).

There are global degrees of freedom associated to the A_{ijk}, one associated to each of the third homology classes of $\mathcal{Z}$. However, these have vanishing Poisson brackets with the other observables, and so just label superselection sectors of the theory[6]

From now on we will assume that $\tilde{\Psi} \neq 0$. Thus we are working only in the sector of the phase space defined by (19) and the nonvanishing of $\tilde{\Psi}$. As there are no other non-vanishing components of the symplectic form, ω, we may invert it on its non-degenerate subspace, to find the Poisson brackets. The only non-vanishing components are,

$$\{A_{\alpha\beta\gamma}(y,z), A_{\delta\epsilon\phi}(y',z')\} = \frac{1}{\tilde{\Psi}(z)} \epsilon_{\alpha\beta\gamma\delta\epsilon\phi} \delta^4(z,z') \delta^6(y,y') \tag{23}$$

We may then integrate to find the Poisson brackets among the surface variables. To do this we may use the product structure to get an embedding of a three surface $\hat{\gamma}^i(\sigma)$ in $\mathcal{Y}$ from every three surface embedding in $\gamma^a(\sigma)$ in $\mathcal{N}$. Similarly, we have an embedding $\hat{\hat{\gamma}}$ in the base $\mathcal{Z}$. Then we have,

$$\{T[\gamma], T[\gamma']\} = \sum_{\sigma_i^*} T[\gamma]T[\gamma'] Int_{\mathcal{Y}}[\hat{\gamma}, \hat{\gamma}'] \frac{1}{\tilde{\Psi}(\hat{\hat{\gamma}}(\sigma_i^*))} \delta^4((\hat{\hat{\gamma}}(\sigma_i^*), (\hat{\hat{\gamma}}'(\sigma_i'^*)) \tag{24}$$

where σ_i^* are the coordinates of points of intersection of the two surfaces and i labels the intersections when there is more than one. Here $Int_{\mathcal{Y}}[\hat{\gamma}, \hat{\gamma}']$ is the intersection number of two three surfaces in the six dimensional space $\mathcal{Y}$, given by

$$Int_{\mathcal{Y}}[\gamma, \gamma'] = \int_\gamma d^3\gamma^{\alpha\beta\gamma}(\sigma) \int_\gamma d^3\gamma'^{\delta\epsilon\phi}(\sigma') \epsilon_{\alpha\beta\gamma\delta\epsilon\phi} \delta^6(\gamma(\sigma), \gamma'(\sigma')) \tag{25}$$

Let us now take the loops to lie entirely in the fibers. In this case we have a four parameter family of loops $\gamma^i(z)$ for each loop γ^i in $\mathcal{Y}$. We then have Poisson brackets

$$\{\phi[\tilde{\gamma}](z), \phi[\tilde{\gamma}'](z')\} = Int_{\mathcal{Y}}[\tilde{\gamma}, \tilde{\gamma}'] \frac{1}{\Psi(z)} \delta^4(z,z') \phi[\tilde{\gamma}](z) \phi[\tilde{\gamma}'](z') \tag{26}$$

[6]Note that this is because we have restricted the theory to the sector defined by (19) and $d = 4$.

Here $Int_{\mathcal{Y}}[\tilde{\gamma}, \tilde{\gamma}']$ is a the oriented intersection number of the homology classes of embeddings of three surfaces in $\mathcal{Y}$.

We may now make use of the result that there is a basis for $H^3(\mathcal{Y}_6)$, consisting of conjugate pairs, (γ_I, π^J), of homology classes of three surfaces such that $Int_{\mathcal{Y}}[\gamma_I, \pi^J] = \delta_I^J$. It is then convenient to choose a corresponding set of canonical fields

$$\phi_I(z) = \phi[\tilde{\gamma}_I](z) \tag{27}$$

and momenta (which are densities in $\mathcal{Z}$),

$$\tilde{\pi}^J(z) = \tilde{\Psi}(z) \int_{\pi^J(z)} A \tag{28}$$

These satisfy the canonical commutation relations,

$$\{\phi_I(z), \tilde{\pi}^J(z')\} = \delta_I^J \delta^4(z, z') \phi_I(z) \tag{29}$$

However, now we must recall the condition (19) which tells us that the observables $\phi[\tilde{\gamma}](z)$ do not actually depend on z. The reason is that because of (19) and the product structure of the manifold,

$$\int_{\tilde{\gamma}(z)} A = \int_{\tilde{\gamma}(z')} A. \tag{30}$$

Thus, this sector of solutions is actually a topological field theory with a finite number of degrees of freedom. To exhibit them explicitly, we may write the conjugate momenta as

$$\Pi^I = \int_{\mathcal{Z}_4} \tilde{\pi}^I \tag{31}$$

so that there is one momentum variable for each conjugate pair of homology classes $H^3(\mathcal{Y}_6)$.

The variables these are conjugate to may just be taken to be

$$\phi_I = \phi_I(z) \tag{32}$$

for any z, as they are all equal. The observables algebra is then,

$$\{\phi_I, \Pi^J\} = \delta_I^J \phi_I \tag{33}$$

Apart from this single commuting variable $\int_{\mathcal{Z}_4} F$ this is the complete observable algebra of the degrees of freedom of the sector defined by (19) with $d = 4$ and $\tilde{\Phi} \neq 0$.

6 Quasi-topological sectors

By counting the full eleven dimensional Chern-Simons theory has local degrees of freedom, but these are not seen in the sector we have just considered. This means that the condition (19) defines too small of a set of solutions to the contraints (9). To free up the local degrees of freedom of the theory we must study a less restrictive set of solutions. It is not easy to study the general case, because of the difficulty of inverting the full symplectic form (13) in the presence of (9). But it is not hard to consider a somewhat less restrictive class of solutions, which has local degrees of freedom.

We will keep the same topological conditions we used in the last section. The base space $\mathcal{Z}$ is then four dimensional and the fibers $\mathcal{Y}$ are six dimensional. But to define the flatness condition we first split the six coordinates y^α into two sets of three

$$y^\alpha = (y^A, y^{\bar{A}}) \tag{34}$$

with $A, \bar{A} = 1, 2, 3$. We will assume that the splitting can be made locally so that there are three surfaces in the "coordinate" homology classes γ_I which are always in the constant y^A surfaces, while there are representatives of the "conjugate" homology classes π^J which are always in the constant $y^{\bar{A}}$ surfaces. For example, we may take $\mathcal{Y} = S^3 \times S^3$, with γ and π wrapping respectively the first and second S^3 in which case $y^{\bar{A}}$ are coordinates of the first S^3 and y^A are coordinates of the second one.

We begin looking for a more general class of solutions by eliminating some degrees of freedom by gauge fixing. First, since there are no local degrees of freedom in the A_{ijk} we will fix them to a constant value A^0_{ijk}. This can be done by a combination of gauge transformations involving $\lambda_{ij}(z)$ and diffeomorphisms of $\mathcal{Z}$. Thus, $\delta A_{ijk} = 0$. Then we may fix many of the mixed components of A_{abc} to vanish by a gauge transformation. For example, if we define $A^A_i = \epsilon^{ABC} A_{BCi}$ we can fix $A^A_i = 0$ by a gauge tranformation. To do this fix a flat background metric δ_{AB} and choose a $\lambda_{Ci} = \delta_{CD}\epsilon^{DEF}\partial_E \rho_{Fi}$ for some ρ_{Ai}. Then under a gauge transformation we have

$$A^{A\prime}_i = 0 = A^A_i + \partial_i \epsilon^{ABC}\lambda_{BC} + \partial^A\partial^E \rho_{Ei} - \nabla^2 \rho_{Ai} \tag{35}$$

This can be solved locally ρ_{Ai} of the form

$$\rho_{Ai} = \frac{1}{\nabla^2}\left(A^A_i + \partial_i \epsilon^{ABC}\lambda_{BC} + \partial^A\partial^E \rho_{Ei}\right) \tag{36}$$

Similarly, we can fix the mixed components $A_{i\bar{A}\bar{B}} = 0$ by using the gauge transoformations parameterized by $\lambda_{\bar{C}i}$.

To proceed further we reduce the degrees of freedom by making an ansatz. We dimensionally reduce by imposing that

$$\partial_A A_{abc} = \partial_{\bar{A}} A_{abc} = 0 \tag{37}$$

so that there is only spatial dependence on the z. Then we set the remaining mixed components to zero by imposing

$$A_{iA\bar{A}} = A_{ijA} = A_{ij\bar{A}} = A_{AB\bar{A}} = A_{A\bar{A}\bar{B}} = 0 \tag{38}$$

The remaining degrees of freedom are the components $\mathcal{A} = \frac{1}{6}A_{ABC}\epsilon^{ABC}$ and $\bar{\mathcal{A}} = \frac{1}{6}A_{\bar{A}\bar{B}\bar{C}}\epsilon^{\bar{A}\bar{B}\bar{C}}$. The non-vanishing components of F_{abcd} are F_{ijkl} and $F_{iABC} = \partial_i \mathcal{A}\epsilon_{ABC}$ and $F_{i\bar{A}\bar{B}\bar{C}} = \partial_i \bar{\mathcal{A}}\epsilon_{\bar{A}\bar{B}\bar{C}}$.

All of the constraints (9) vanish automatically under this dimensional reduction except,

$$H^{ij} = \epsilon^{ijkl}\partial_k \mathcal{A}\partial_l \bar{\mathcal{A}} = 0. \tag{39}$$

The symplectic form, in the presence of the dimensional reduction is then,

$$\omega(\delta_1 A, \delta_2 A) = \int \Big(\delta_1 \mathcal{A} \wedge \delta_2 \bar{\mathcal{A}}\tilde{\Phi} - (1 \rightarrow 2)\Big) \tag{40}$$

We can now invert to find the Poisson brackets,

$$\{\mathcal{A}(y,z), \bar{\mathcal{A}}(y',z')\} = \frac{1}{\tilde{\Phi}}\delta^6(y,y')\delta^4(z,z') \tag{41}$$

Once we have these we can write the algebra for the gauge invariant observables. The fiber observables algebra works as before, except that now the $\phi_I(z)$ and $\tilde{\pi}^J$ do depend on z. But we have to remember also the remaining constraint (39). One set of solutions follows if we set $\partial_i \mathcal{A} = 0$. Then the remaining degrees of freedom are the

$$\phi_I(z) = e^{\int_{\gamma^I} \bar{\mathcal{A}}\epsilon_{\bar{A}\bar{B}\bar{C}}} \tag{42}$$

which become functions on $\mathcal{Z}$.

By choosing representatives of the π^I that only involve the $\mathcal{A}$ we then find the observables algebra,

$$\{\phi_I(z), \Pi^J\} = \delta_I^J \phi_I(z) \tag{43}$$

Thus, by dimensional reduction we have arrived at a reduction of the theory that has a gauge invariant observables algebra. We see that the

resulting structure is a new kind of combination of a conventional and topological quantum field theory, which we may call quasi-topological. We have an infinite set of coordinate observables, who are defined as local fields on a lower dimensional submanifold. Each of them is connected with homotopy classes of the fiber spaces. In this sense we have something like a field theory of topological field theories[7]. The structure of the conjugate momenta are even more unusual, as we have only a finite set of distinct momenta who are associated to homotopy classes of the fibers.

It is clear that this kind of structure has arisen because of we have considered a truncation of the theory defined by the dimensional reduction given by (37) and (38). The full set of solutions is likely to be even more intricate. Steps towards the construction of the general canonical formalism are given in [39]. For the present we confine ourselves to a discussion of the quantum theory associated with this quasi-topological sector.

7 Quantum theory of the $4 + 6$ sector

We may now proceed to a sketch of the quantum theory associated to the sector of the theory we have just defined. In fact, since the topology of $\mathcal{M}$ is fixed before the quantization we have one theory for each compact six manifold $\mathcal{Y}$. In the last section we concluded that if the homology has a basis of N pairs (γ_I, π^I), we have an observables algebra (29) consisting of N pairs of fields and momenta on the four manifold $\mathcal{Z}$, given by (43).

We are interested in a representation of (43) on which we can also have a representation of $Diff(\mathcal{Z})$, so we can mod out by the diffeomorphisms to find the physical states. We can thus not use a standard Fock representation. As in the construction of the loop representation we have to first construct a non-separable state space on which $Diff(\mathcal{Z})$ has an unbroken action. To do this we consider the $\phi_I(z)$ to be creation operators that create an excitation, which corresponds to a surface in the I'th homology class in the fiber $\mathcal{Y}$ over the point $z \in \mathcal{Z}$. That is we take the continuous product of Fock spaces over each point z. Thus, we have a vacuum state $|0>$ defined by $\tilde{\pi}(z)|0>= 0$. We then define states by occupation numbers at each point z. Thus, a general state consists of a finite list of of excitations $|(I_1, z_1)....(I_n, z_n)>$ where each pair (I, z) represents a surface in the I'th homology created at

[7] This kind of structure was found for the five dimensional abelian Chern-Simons theory by [35].

the point z. The action of the field operators is then to add excitations

$$\hat{\phi}_I(z)|(I_i, z_i) >= |(I, z) \cup (I_i, z_i) > \tag{44}$$

The Hilbert space is then simply

$$\mathcal{H}_{kin} = \prod_z \mathcal{H}_z \tag{45}$$

where a basis for the Hilbert space at each point $\mathcal{H}_z$ is labeled by $|n_1, ..., n_N >$, where the n_i's are the occupation numbers in each of the homology classes $\tilde{\gamma}_I$. A state is then given by

$$|\Psi >= \otimes_z |\psi, z > \tag{46}$$

where $|\psi, z > \in \mathcal{H}_z$. The inner product is then simply the product,

$$< \Psi|\Psi' >= \prod_z < \psi, z|\psi', z > \tag{47}$$

The normalizable states are those in which there are only a finite number of excitations, so that only a finite number of the factors in (47) differ from one.

The action of the conjugate momentum variables is defined by,

$$\hat{\Pi}^J|(I_1, z_1)....(I_n, z_n) >= \delta^J_{I_i}|(I_1, z_1)..(I_{i-1}, z_{i-1})(I_{i+1}, z_{i+1})..(I_n, z_n) > \tag{48}$$

i.e. the operator acts to remove the excitations in $\tilde{\gamma}_J$.

The space (45) is non-separable. But it is easy to mod out by diffeomorphisms. To do this we may define a unitary representation of $Diff(\mathcal{Z})$ on $\mathcal{H}_{kin}$. Given $\phi \in Diff(\mathcal{Z})$ we may define

$$U(\phi)|(I_i, z_i) >= |(I_i, \phi^{-1} \cdot z_i) > \tag{49}$$

It is straightforward to check that this operator is unitary. As in the case of the loop representation, there are no anomalies of the diffeomorphism group. Diffeomorphism invariant states are then defined by

$$\Psi[(I_i, z_i)] =< \Psi|(I_i, z_i) >= \Psi[(I_i, \phi^{-1} \cdot z_i)] \tag{50}$$

The resulting diffeomorphism invariant Hilbert space $\mathcal{H}^{diffeo}$ is labeled simply by a basis of states corresponding to excitations at distinct points, with

no labelings as to where the points are in $\mathcal{Z}$. Thus, a basis is given by a finite set of P lists

$$|(n_i)_1, ..., (n_i)_P > \tag{51}$$

where each list $n_i = (n_1, ..., n_N)$ consists of occupation numbers for the N homology classes $\tilde{\gamma}_I$. Thus, $\mathcal{H}^{diffeo}$ has a separable basis. A non-separable Hilbert space was just needed as a technical device at the kinematical level, as in non-perturbative quantum gravity. Diffeomorphism invariant observables exist, such as the number operators

$$N^I \equiv \int_{\mathcal{Z}} \phi_I(z) p^I \tag{52}$$

(where no sum on I is taken.) More complicated operators may be easily constructed, which measure how many points there are at which there is a certain pattern of excitations.

As there are no interesting diffeomorphism classes of sets of points, the diffeomorphism invariant quantum theory is in this case rather boring. We see that there are local degrees of freedom in the four dimensional manifold $\mathcal{Z}$ that do correspond to three dimensional surfaces wrapped around various homology classes of the six manifold $\mathcal{Y}$. But there are no interesting relationships or interactions among them. The structure of the other cases in which $d > 4$ are more interesting, as there are non-trivial extended structures in both $\mathcal{Z}$ and $\mathcal{Y}$. These will be discussed elsewhere.

8 Conclusions

To summarize, we have found a sector of the solution space of the theory which corresponds to bundles defined by fibering six dimensional compact manifolds $\mathcal{Y}$ over a four manifold $\mathcal{Z}$. The degrees of freedom are a canonically conjugate pair of fields $(\phi_I(z), \tilde{\Pi}^J)$ on $\mathcal{Z}$ corresponding to a basis of the third homology of $\mathcal{Y}$. These have a quasi-topological structure in that the $\phi_I(z)$ are local fields on $\mathcal{Z}$ while the momenta Π^I depend only on the homology classes. The observables algebra is given entirely in terms of the intersection numbers of the surfaces in $\mathcal{Y}$.

Thus, we have achieved our goal of finding a sector of the theory corresponding to a natural compactification of the theory in which the geometry of the compactified directions is entirely represented by an algebra of functions on the base manifold.

Furthermore, we have constructed the quantum theory associated to this sector of the theory, and discovered it consists of diffeomorphism classes (in the four dimensional manifold $\mathcal{Z}$) of excitations of wrappings of three surfaces around homology classes in the six dimensional manifold $\mathcal{Y}$.

ACKNOWLEDGEMENTS

I am especially grateful to Eli Hawkins for seeing an error in the original version of this paper. In addition, I would like to thank John Baez, Louis Crane, Fotini Markopoulou and Andrew Strominger for discussions and encouragement. Comments on the manuscript by Shyamoli Chaudhuri and Juan Maldecena were also very helpful.This work was supported by NSF grant PHY-9514240 to The Pennsylvania State University and a NASA grant to The Santa Fe Institute.

References

[1] M. J. Duff, P. Howe, T. Inami, K.S. Stelle, *Superstrings in D=10 from supermembranes in D=11*, Phys. Lett. B191 (1987) 70; M. J. Duff, J. X. Lu *Duality rotations in membrane theory*, Nucl. Phys. B347 (1990) 394; M. J. Duff, R. Minasian, James T. Liu *Eleven dimensional origin or string/string duality: a one-loop test*, Nucl. Phys. B452 (1995) 261; C. Hull and P. K. Townsend, *Unity of superstring dualities*, Nucl. nPhys. B438 (1995) 109, hep-th/9410167; *Enhanced gauge symmetries in superstring theories* Nucl. Phys. B451 (1995) 525, hep-th/9505073; P. K. Townsend, *The eleven dimensional supermembrane revisited* Phys. Lett. B350 (1995) 184, hep-th/9501068; *String-membrane duality in seven dimensions*, Phys. Lett. 354B (1995) 247, hep-th/9504095.

[2] E. Witten, *String theory dynamics in various dimensions* Nucl. Phys. B443 (1995) 85, hep-th/9503124.

[3] T. Banks, W. Fishler, S. H. Shenker, L. Susskind, *M theory as a matrix model: a conjecture* hep-th/9610043.

[4] M. Atiyah, *Topological quantum field theory* Publ. Math. IHES 68 (1989) 175; *The Geometry and Physics of Knots*, Lezion Lincee (Cambridge University Press, Cambridge,1990); G. Segal, *Conformal field theory* oxford preprint (1988).

[5] A. Joyal and R. Street, *Braided monoidal categories*, Macqurie Mathematics Reports, no. 860081 (1986); *The geometry of tensor calculus I*Adv. Math. 88 (1991) 55; D. N. Yetter, *Quantum groups and representations of monoidal categories*, Math. Proc. Cam. Phil. Soc. 108 (1990) 261; N. Yu. Reshetikhin, *Quasitriangular Hopf algebras and invariants of tangles*, Lenningrad Math. J. 1 (1990) 491; V. Chari and A. Pressley, *A guide to Quantum Groups*, Cambridge University Press, 1994. For the generalization of spin networks to moniodal categories, see Chapter 5.

[6] L. Crane and D. Yetter, *On algebraic structures implicit in topological quantum field theories*, Kansas preprint, (1994); in *Quantum Topology* (World Scientific, 1993) p. 120; L. Crane and I. B. Frenkel, J. Math. Phys. 35 (1994) 5136-54;

[7] E. Witten, *Quantum field theory and the Jones Polynomial* Commun. Math. Phys. 121 (1989) 351; V Turaev O Viro: Topology 31 (1992) 865; L. Crane, Commun. Math. Phys. 135 (1991) 615; Phys. Lett. B 259 (1991) 243; E. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory* Nucl. Phys. B 300 (1988) 360; G. Moore and N. Seiberg, *Classical and quantum conformal field theories* Comun. Math. Phys. 123 (1988) 177; G. Moore and N. Yu. Reshetikhin, *A comment on quantum group symmetry in conformal field theory*, Nucl. Phys. B 328 (1989) 557; N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Commun. Math. Phys. 127 (1990) 1; *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. 103 (1991) 547.

[8] J. D. Bekenstein, Lett. Nuovo Cim 11 (1974) 467.

[9] G. 'tHooft, *Dimensional reduction in quantum gravity* gr-qc/9310006.

[10] L. Susskind, *The world as a hologram*, hep-th/9409089, J. Math. Phys. (1995) ; *Strings, black holes and Lorentz contractions* hep-th/9308139; Phys. Rev. Lett. 71 (1993) 2367; Phys. Rev. D 49 (1994) 6606; D 50 (1994) 2700; L. Susskind and P. Griffin, *Partons and black holes* hep-th/9410306;

[11] L. Smolin, *Linking topological quantum field theory and nonperturbative quantum gravity* gr-qc/9505028, J. Math. Phys. 36 (1995) 6417.

[12] R. Penrose, *Theory of quantized directions* unpublished manuscript; in *Quantum theory and beyond* ed T Bastin, Cambridge U Press 1971; in *Advances in Twistor Theory*, ed. L. P. Hughston and R. S. Ward, (Pitman,1979) p. 301; in *Combinatorial Mathematics and its Application* (ed. D. J. A. Welsh) (Academic Press,1971).

[13] L. Kauffman and S. Lins *Tempereley-Lieb Recoupling Theory and Invariants of 3-Manifolds* Princeton University Press, 1994, and references therein.

[14] C. Rovelli and L. Smolin, "Spin networks and quantum gravity" gr-qc/9505006, Physical Review D 52 (1995) 5743-5759.

[15] J. Baez, Spin networks in gauge theory, Adv. Math. **117** (1996), 253-272, gr-qc/941107; Spin networks in nonperturbative quantum gravity, in *The Interface of Knots and Physics*, ed. Louis Kauffman, American Mathematical Society, Providence, Rhode Island, 1996, gr-qc/9504036.

[16] L Smolin: in *Quantum Gravity and Cosmology*, eds J Pérez-Mercader *et al*, World Scientific, Singapore 1992.

[17] T. Jacobson and L. Smolin, Nucl. Phys. B 299 (1988); C Rovelli L Smolin: Phys Rev Lett 61 (1988) 1155; Nucl Phys B133, 80 (1990).

[18] C. Rovelli and L. Smolin *Discreteness of area and volume in quantum gravity* Nuclear Physics B 442 (1995) 593. Erratum: Nucl. Phys. B 456 (1995) 734.

[19] L. Smolin, *The future of spin networks* gr-qc/9702030.

[20] A. Ashtekar and C. J. Isham, Class and Quant Grav 9 (1992) 1069; A Ashtekar J Lewandowski D Marlof J Mourãu T Thiemann: "Quantization of diffeomorphism invariant theories of connections with local degrees of freedom", gr-qc/9504018, JMP 36 (1995) 519; A. Ashtekar and J. Lewandowski, "Quantum Geometry I: area operator" gr-qc/9602046; J. Lewandowski, "Volume and quantization" gr-qc/9602035.

[21] T. Thiemann, Quantum spin dynamics I, Harvard preprint (1996), gr-qc/9606089.

[22] T. Thiemann, Quantum spin dynamics II, Harvard preprint (1996), gr-qc/9606090.

[23] C Rovelli L Smolin: Phys Rev Lett 72 (1994) 446

[24] R. Borissov, *Graphical evolution of spin network states*, gr-qc/9605

[25] R. Borissov, Ph.D. thesis, Temple, (1996).

[26] R. Borissov, R. de Pietro and C. Rovelli, preprint in preparation, (1997).

[27] L. Smolin, *The classical limit and the form of the hamiltonian constraint in non-pertubative quantum gravity* CGPG preprint, gr-qc/9609034.

[28] F. Markopoulou and L. Smolin, *Causal evolution of spin networks* gr-qc/9702025. CGPG 97/2-1; F. Markopoulou, *Dual formulation of spin network evolution* preprint, March 1997.

[29] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. B76 (1978) 409.

[30] J. Polchinski *Dirichlet branes and Ramond-Ramond charges*, Phys. Rev. Lett. 75 (1995) 4724; for a review see J. Polchinski, S. Chaudhuri and C. V. Johnson, *Notes on D-branes* hep-th/9602052.

[31] See, for example, A. Sen hep-th/9504147; F. Larsen and F. Wilczek, hep-th/9511064; A Strominger and C. Vafa, Phys. Lett. B379 (1996) 99, hep-th/9601029; C. Callen and J. Maldecena, Nucl. Phys. B 475 (1996) 645; hep-th/9602043; A. W. Peet, hep-th/9506200; J.Maldacena, *Black holes i string theory* Ph.D. thesis, and references contained theirin.

[32] T. Jacobson and L. Smolin, Phys. Lett. B196 (1987) 39; Class. Quan. Grav. 5 (1988) 583; J. Samuel, Pramana-J Phys. 28 (1987) L429.

[33] R. Capovillia, J. Dell and T. Jacobson, Phys. Rev. Lett. 63 (1989) 2325.

[34] R. Floreanini and R. Percacci, Phys. Lett. B224 (1989) 291-294; B231:119-124,1989.

[35] V.V. Fock, N.A. Nekrasov, A.A. Rosly, K.G. Selivanov *What we think about the higher dimensional Chern-Simons theories* (Moscow, ITEP). ITEP-91-70, July 1991. 7pp. in Sakharov Conf.1991:465-472 (QC20:I475:1991)

[36] Max Banados, Luis J. Garay and Marc Henneaux, Nucl.Phys.B476:611-635,1996, hep-th/9605159; Phys.Rev.D53:593-596,1996, hep-th/9506187

[37] R. Gambini, Phys. Lett. B 171 (1986) 251; P. J. Arias, C. Di Bartolo, X. Fustero, R. Gambini and A. Trias, Int. J. Mod. Phys. A 7 (1991) 737.

[38] L. Smolin, Phys. Rev. D 49 (1994) 4028.

[39] M. Banados, M. Henneaux, C. Iannuzzo and C. M. Viallet, *A note on the gauge symmetries of pure Chern-Simons theory with p-form gauge fields* gr-qc/9703061.

[40] S. Melosch and H. Nicolai —it New canonical variables for d=11 supergravity hel-th/9709227.